CHAPTER 2

COMPUTATIONS WITH LOGARITHMS

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Multiply and divide numbers using logarithms.
2. Compute the power of a number and the root of a number using logarithms.
3. Apply the laws for logarithms to algebraic operations and to problem solving.

INTRODUCTION

In this chapter additional mention of the Laws for Logarithms will be given followed by algebraic operations and applications using logarithms.

Laws for Powers and Roots are listed in table 2-1 for reference and review.

Table 2-1.—Laws For Powers and Roots

APPLICATION	LAW	EXAMPLE
Multiplication	$a^n a^m = a^{n+m}$	$3^4 \cdot 3^2 = 3^{4+2} = 3^6$
Division	$\frac{a^n}{a^m} = a^{n-m}$	$\frac{3^4}{3^2} = 3^{4-2} = 3^2$
Power Raised to a Power	$(a^n)^m = a^{nm}$	$(3^4)^2 = 3^{4 \cdot 2} = 3^8$
Product Raised to a Power	$(ab)^n = a^n b^n$	$(3 \cdot 5)^4 = 3^4 \cdot 5^4$
Quotient Raised to a Power	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, if $b \neq 0$	$\left(\frac{3}{5}\right)^4 = \frac{3^4}{5^4}$
Negative Power	$a^{-n} = \frac{1}{a^n}$	$3^{-4} = \frac{1}{3^4}$
Square Root	$\sqrt{a} = a^{1/2}$	$\sqrt{3} = 3^{1/2}$
nth Root	$\sqrt[n]{a} = a^{1/n}$	$\sqrt[4]{3} = 3^{1/4}$
nth Root of a Power	$\sqrt[n]{a^m} = a^{m/n}$	$\sqrt[4]{3^2} = 3^{2/4}$

All calculations by means of logarithms in this chapter use 10 as the base. By the convention established in chapter 1, the expression log A is understood to mean the base 10 logarithm of A.

Computations using logarithms will be evaluated to four significant digits to correspond with interpolation procedures introduced in chapter 1. All the digits of an approximate number except zeros, which serve only to fix the position of the decimal point, are called *significant digits.*

MULTIPLICATION

Law 1. *The logarithm of a product is equal to the sum of the logarithms of the factors;* that is,

$$\log AB = \log A + \log B$$

EXAMPLE: Use logarithms to find the product of 386 × 254 to four significant digits.

SOLUTION:

Exponential solution:

$$\begin{aligned} 386 &= 10^{2.5866} \\ 254 &= 10^{2.4048} \\ 386 \times 254 &= 10^{2.5866} \times 10^{2.4048} \\ &= 10^{(2.5866 + 2.4048)} \\ &= 10^{4.9914} \end{aligned}$$

Logarithmic solution:

$$\begin{aligned} \log 386 &= 2.5866 \\ \log 254 &= 2.4048 \\ \log (386 \times 254) &= \log 386 + \log 254 \\ &= 2.5866 + 2.4048 \\ &= 4.9914 \end{aligned}$$

To find the antilogarithm of 4.9914, we will interpolate:

$$
\begin{array}{ccc}
 & \text{NUMBER} \qquad \text{MANTISSA} & \\
.01 & \left[\begin{array}{l} x\left[\begin{array}{cc} 9.80 & .9912 \\ & \\ ? & .9914 \end{array}\right] .0002 \\ \\ \quad\begin{array}{cc} 9.81 & .9917 \end{array} \end{array}\right] & .0005
\end{array}
$$

$$x = \frac{.0002}{.0005}(.01) = \frac{2}{5}(.01) = .004$$

Therefore, antilog 4.9914 = 98,040, which is very close to the actual calculated value of 98,044.

The exponential solution shown in the example is not a part of normal calculations involving logarithms. It was shown in this first example problem solely for the purpose of reemphasizing the relationship between exponents and logarithms.

NOTE: The logarithmic value of 98,040 is not the same as the actual value of 98,044, because the logarithmic table values (used in this book) are only significant to four digits. The larger the table values, the closer the logarithmic value is to the actual value.

EXAMPLE: Use logarithms to find the product of (126)($-$33) to four significant digits.

SOLUTION: Recall from chapter 1 that negative numbers do not have logarithms. In using logarithms to solve problems that involve negative numbers, we first determine the sign of the final answer. After this sign is determined, calculations are performed as if all numbers are positive, and then the predetermined sign is applied to the answer.

In our example, dealing first with signs only, we determine the answer to be negative; that is, $(+)(-) = (-)$. At this point the problem can be restated: Use logarithms to find the product of $-(126 \times 33)$.

$$
\begin{aligned}
\log (126 \times 33) &= \log 126 + \log 33 \\
&= 2.1004 + 1.5185 \\
&= 3.6189
\end{aligned}
$$

and by interpolation,

$$\text{antilog } 3.6189 = 4{,}158$$

Therefore, $(126)(-33) = -4{,}158$. This value is the same as the actual value.

EXAMPLE: Use logarithms to find the product of $1.73 \times 0.0024 \times 0.08$ to four significant digits.

SOLUTION:

$$
\begin{aligned}
& \log\,(1.73 \times 0.0024 \times 0.08) \\
&= \log 1.73 + \log 0.0024 + \log 0.08 \\
&= 0.2380 + (7.3802 - 10) + (8.9031 - 10) \\
&= 16.5213 - 20 \\
&= 6.5213 - 10
\end{aligned}
$$

To find antilog $(6.5213 - 10)$, we will interpolate:

$$
\begin{array}{c}
\qquad\quad \text{NUMBER} \qquad\quad \text{MANTISSA} \\
.01\left[\begin{array}{c}
x\left[\begin{array}{cc} 3.32 & .5211 \\ ? & .5213 \end{array}\right].0002 \\
\begin{array}{cc} 3.33 & .5224 \end{array}
\end{array}\right].0013
\end{array}
$$

$$x = \frac{.0002}{.0013}(.01) = \frac{2}{13}(.01) = .002$$

So, antilog $(6.5213 - 10) = 0.0003322$. The logarithmic value is again the same as the actual value to four significant digits.

PRACTICE PROBLEMS:

Use logarithms to find the product of the following to four significant digits:

1. $(53)(-76)(-0.021)(153)$
2. $1.02 \times 10^{9} \times 4.76 \times 10^{-3}$

3. (0.00432)(−0.00106)(15)

4. $0.102 \times 103.5 \times 76.2$

ANSWERS:

1. 12,940

2. 4,856,000

3. -6.869×10^{-5}

4. 804.4

DIVISION

Law 2. *The logarithm of a quotient is equal to the difference of the logarithms of the dividend and the divisor;* that is,

$$\log \frac{A}{B} = \log A - \log B$$

EXAMPLE: Find the quotient of 37.4/1.7 by use of logarithms to four significant digits.

SOLUTION:

$$\begin{aligned} \log (37.4/1.7) &= \log 37.4 - \log 1.7 \\ &= 1.5729 - 0.2304 \\ &= 1.3425 \end{aligned}$$

and

$$\text{antilog } 1.3425 = 22.01$$

and 22.00 is the actual value.

EXAMPLE: Use logarithms to find the quotient of 16.3/0.008 to four significant digits.

SOLUTION:

$$\log (16.3/0.008) = \log 16.3 - \log 0.008$$
$$= 1.2122 - (7.9031 - 10)$$

To prevent the complication of subtracting a larger characteristic (7) from a smaller characteristic (1), we add 10 to and subtract 10 from the logarithm of the dividend. Note that this does not change the value of the logarithm. Thus,

$$\begin{array}{rl} \log 16.3 = & 11.2122 - 10 \\ - \log 0.008 = & \underline{\ 7.9031 - 10} \\ & \ 3.3091 \end{array}$$

and

$$\text{antilog } 3.3091 = 2{,}038$$

Therefore, the logarithmic value of 16.3/0.008 is 2,038, while the actual value is 2,037.5.

PRACTICE PROBLEMS:

Use logarithms to solve the following problems to four significant digits:

1. 635.6/25.4

2. 0.26/0.061

3. 0.126/0.00542

4. 874/26.3

ANSWERS:

1. 25.03

2. 4.263

3. 23.25

4. 33.23

POWERS

Law 3. *The logarithm of a number raised to a power is equal to the exponent times the logarithm of the number;* that is,

$$\log A^n = n \log A$$

EXAMPLE: Use logarithms to find the value of $(18.53)^5$ to four significant digits.

SOLUTION:

$$\begin{aligned}\log (18.53)^5 &= 5 \log 18.53 \\ &= 5(1.2679) \\ &= 6.3395\end{aligned}$$

and

$$\text{antilog } 6.3395 = 2{,}185{,}000$$

So the logarithmic value of $(18.53)^5$ is 2,185,000, while the actual value is 2,184,626.

ROOTS

Law 4. *The logarithm of the nth root of a number is equal to the logarithm of the number divided by n, the index of the root;* that is,

$$\log \sqrt[n]{A} = \frac{1}{n} \log A$$

EXAMPLE: Use logarithms to find the value of $\sqrt[5]{327.6}$ to four significant digits.

SOLUTION:

$$\log \sqrt[5]{327.6} = \frac{1}{5} \log 327.6$$

$$= \frac{1}{5}(2.5153)$$

$$= 0.5031$$

where

$$\text{antilog } 0.5031 = 3.185$$

which is the logarithmic value and the actual of $\sqrt[5]{327.6}$ to four significant digits.

When a logarithm with a negative characteristic is to be divided, adding and subtracting a number that will, after dividing, leave a minus 10 at the right is advisable. This is done to keep the logarithm in standard form.

EXAMPLE: Find the value of $\sqrt[5]{0.0018}$ to four significant digits using logarithms.

SOLUTION:

$$\log \sqrt[5]{0.0018} = \frac{1}{5} \log 0.0018$$

$$= \frac{1}{5}(7.2553 - 10)$$

To keep a minus 10 in the final logarithm, we must add and subtract 40 before dividing. Thus,

$$\log \sqrt[5]{0.0018} = \frac{1}{5}(47.2553 - 50)$$

$$= 9.4511 - 10$$

where

$$\text{antilog } (9.4511 - 10) = 0.2826$$

Therefore, the logarithmic value of $\sqrt[5]{0.0018}$ is 0.2826, while the actual value is 0.2825 to four significant digits.

PRACTICE PROBLEMS:

Evaluate the following to four significant digits using logarithms:

1. $(3.276)^3$
2. $(0.00468)^2$
3. $\sqrt[6]{0.00867}$
4. $\sqrt[5]{237.7}$

ANSWERS:

1. 35.15
2. 0.00002190
3. 0.4532
4. 2.987

ALGEBRAIC OPERATIONS

This chapter has demonstrated the use of logarithms in numerical calculations. *Practical applications in many fields involve calculations including algebraic expressions in which logarithms are useful. In these problems both the laws for algebra and the laws for logarithms in algebraic operations are valid.* For example,

$$\log (x + 2)(x + 5) = \log (x + 2) + \log (x + 5)$$

EXAMPLE: Solve for x in the equation

$$\log (x^2 - 5x - 6) - \log (x + 1) = 1.$$

NOTE: $\log 10 = 1$.

SOLUTION:

$$\log \frac{x^2 - 5x - 6}{x + 1} = \log 10$$

$$\log \frac{(x - 6)(x + 1)}{(x + 1)} = \log 10$$

$$\log (x - 6) = \log 10$$

$$x - 6 = 10$$

$$x = 16$$

EXAMPLE: Solve for x to four significant digits using logarithms:

$$26^x = 195$$

SOLUTION: Take the logarithm of both sides of the equation:

$$\log 26^x = \log 195$$

$$x \log 26 = \log 195$$

$$x = \frac{\log 195}{\log 26}$$

$$= \frac{2.2900}{1.4150}$$

$$= 1.618$$

In complicated problems we may not be able to solve for the unknown as directly as we did in the previous example. In that case we can continue to use our knowledge of logarithms. For instance, return to the step where $x = 2.29/1.415$; take the logarithm of both sides of the equation

$$\log x = \log \frac{2.29}{1.415}$$

$$= \log 2.29 - \log 1.415$$

$$= 0.3598 - 0.1508$$

$$= 0.2090$$

Now take the antilogarithm of both sides:

$$x = \text{antilog } 0.2090$$
$$= 1.618$$

The logarithmic and actual values of x are both 1.618 to four significant digits, so $26^{1.618} \approx 195$.

EXAMPLE: Solve for x using logarithms:

$$x^{3/2} = 729$$

SOLUTION:

$$\log x^{3/2} = \log 729$$
$$(3/2) \log x = \log 729$$
$$\log x = (2/3) \log 729$$
$$= (2/3)(2.8627)$$
$$= 1.9085$$
$$x = \text{antilog } 1.9085$$
$$= 81$$

The logarithmic and actual values are both 81, so $81^{3/2} = 729$.

PRACTICE PROBLEMS:

Use logarithms to solve for x to four significant digits in the following:

1. $1.7^x = 3.1$
2. $x^{8/3} = 6.35$

ANSWERS:

1. 2.133
2. 2.000

APPLICATIONS

The use of logarithms can simplify the solution of many problems encountered in mathematics, science, and engineering. Applying the operations described in this chapter can reduce many complicated equations to addition and subtraction problems.

EXAMPLE: Find the volume of a circular cone having a height of 3.71 inches and a base radius of 2.71 inches.

SOLUTION: The formula for volume of a circular cone is

$$v = \frac{\pi r^2 h}{3}$$

where v is volume, r is radius, h is height, and π (pi) is equal to 3.142 to four significant digits.

Take the logarithm of both sides of the equation as the first step in the solution and continue with the Laws for Logarithms.

$$\begin{aligned}
\log v &= \log \left(\frac{\pi r^2 h}{3}\right) \\
&= \log \pi + \log r^2 + \log h - \log 3 \\
&= \log (3.142) + 2 \log (2.71) + \log (3.71) - \log (3) \\
&= 0.4972 + 2(0.4330) + 0.5694 - 0.4771 \\
&= 1.4555 \\
v &= \text{antilog } 1.4555 \\
&= 28.54
\end{aligned}$$

The volume of the circular cone using logarithms to four significant digits is 28.54 inches cubed, while the actual value to four significant digits is 28.53 inches cubed.

Many electronics problems can be simplified by using logarithms. Some electronics problems include common logarithms in the basic formulas. An example of a formula that includes a logarithmic expression is the formula for finding gain in decibels, where

$$\text{decibels} = 10 \log \frac{P_1}{P_2}$$

Engineering and electronics problems frequently deal with numbers in the millions and decimal fractions in the millionths. These values are easily expressed as exponentials to the base 10, and common logarithms are then a natural and convenient means of simplifying these problems.

EXAMPLE: $X_C = \frac{1}{2\pi fC}$ is a formula used to analyze alternating current circuits. Use logarithms to find X_C to four significant digits, if $f = 22{,}000{,}000$ Hertz and $C = 1.5 \times 10^{-9}$ farads.

SOLUTION: Taking the logarithm of both sides of the formula $X_C = \frac{1}{2\pi fC}$ gives

$$\begin{aligned}
\log X_C &= \log 1 - \log (2\pi fC) \\
&= 0 - [\log 2 + \log 3.142 + \log (2.2 \times 10^7) \\
&\quad + \log (1.5 \times 10^{-9})] \\
&= -\ [0.3010 + 0.4972 + 7.3424 + (1.1761 - 10)] \\
&= -\ (9.3167 - 10) \\
&= -\ 9.3167 + 10 \\
&= 0.6833 \\
X_C &= \text{antilog } 0.6833 \\
&= 4.823 \text{ ohms}
\end{aligned}$$

PRACTICE PROBLEMS:

Use logarithms to solve for the numerical value of the unknown in the following problems to four significant digits:

1. Find the volume (v) of a sphere, given the formula

$$v = \frac{4\pi r^3}{3}$$

where the radius (r) is 7.59 and π is 3.142.

2. Find the value of I in the formula

$$P = I^2R$$

when $P = 217$ and $R = 550{,}000$.

ANSWERS:

1. 1,831

2. 0.01986

SUMMARY

The following are the major topics covered in this chapter:

1. **Significant digits:** All the digits of an approximate number except zeros, which serve only to fix the position of the decimal point, are called *significant digits.*

2. **Multiplication:**

   Law 1. *The logarithm of a product is equal to the sum of the logarithms of the factors.*

3. **Division:**

   Law 2. *The logarithm of a quotient is equal to the difference of the logarithms of the dividend and the divisor.*

   To prevent the complication of subtracting a larger characteristic from a smaller characteristic, add 10 to and subtract 10 from the logarithm of the dividend.

4. **Powers:**

   Law 3. *The logarithm of a number raised to a power is equal to the exponent times the logarithm of the number.*

5. **Roots:**

   Law 4. *The logarithm of the nth root of a number is equal to the logarithm of the number divided by n, the index of the root.*

   When a logarithm with a negative characteristic is to be divided, add and subtract a number that will, after dividing, leave a minus 10 at the right. This is done to keep the logarithm in standard form.

6. **Algebraic operations:** Practical applications in many fields involve calculations in which logarithms are useful. In these problems both the laws for algebra and the laws for logarithms in algebraic operations are valid.

7. **Applications:** The use of logarithms can simplify the solution of many problems encountered in mathematics, science, and engineering. Applying operations can reduce many complicated equations to addition and subtraction problems.

ADDITIONAL PRACTICE PROBLEMS

Use logarithms to solve the following problems to four significant digits:

1. $\dfrac{(-46.3)(189)}{(-2.13)}$

2. $\dfrac{(815)}{(7.95)^4}$

3. $(-2.46)^3\ (1.11)^5$

4. $\sqrt[5]{\dfrac{(49.9)(5.00)}{(0.0348)}}$

5. Solve for x:

$$5 \cdot 4^x = 6 \cdot 3^x$$

HINT: $\dfrac{4^x}{3^x} = \left(\dfrac{4}{3}\right)^x$

6. $H.P. = \dfrac{1.28apv^2}{1{,}100}$ is a formula used in aeronautics.

Find $H.P.$ when $p = 0.003$, $a = 5.5$ and $v = 253.7$.

7. The chemist defines the pH (hydrogen potential) of a solution by

$$pH = \log \frac{1}{[H^+]}$$

where $[H^+]$ is a numerical value for the concentration of hydrogen ions in an aqueous solution in moles per liter. Calculate the pH of a solution whose hydrogen ion concentration is 3.7×10^{-6}.

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. 4,109
2. 0.2040
3. -25.07
4. 5.903
5. 0.6336
6. 1.236
7. 5.4318